

DENIAL LOGIC

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ABSTRACT. Denial Logic is the logic of an agent whose justified beliefs are false, who cannot avow his own propositional attitudes or believe tautologies, but who can believe contradictions. Denial Logic DL is defined as justification logic JL together with the Denial axiom $t : E \rightarrow \neg E$ and the Evidence Pairing axiom $s : D \wedge t : E \rightarrow [s \& t] : D \wedge E$. Using Artemov’s natural JL semantics, in which justifications are interpreted as sets of formulas, we provide an inductive construction of models of DL, and show that DL is sound and complete. Some notions developed for JL, such as constant specifications and internalization, are inconsistent with DL. In contrast, we define negative constant specifications, which can be used to model agents in DL with justified false beliefs. Denial logic can therefore be relevant to philosophical skepticism. We define coherent negative constant specifications for DL to model a Putnamian brain in a vat with the justified false belief that it is not a brain in a vat, and prove a “Blue Pill” theorem, which produces a model of JL in which “I am a brain in a vat” is false. We extend DL to the multi-modal logic $DL \oplus_{JL} LP$ to model envatted brains who can justify and check tautologies and avow their own propositional attitudes. Denial Logic was inspired by online debates over anthropogenic global warming.

Keywords justification logic; logic of proofs; modal and epistemic logic; skepticism.

INTRODUCTION

This paper is a contribution to the study of logics of skepticism. Our setting is Justification Logic JL, the minimal logic for a family of logics that includes the Logic of Proofs LP, a Hilbert-style logical system extending classical and intuitionistic propositional logic, with additional propositional types of the form $(t : A)$, read as “term t is justification for A ” [3, 5]. LP and the related broader class of justification logics are, in a precise sense, refinements of modal logics, including K, K4, K45, KD45, T, S4 and S5 [3, 4, 5, 6, 7]. Semantics for these systems include Kripke–Fitting models, Mkrtychev models, and arithmetical provability semantics [20, 30, 3]. Recently, Artemov provided Justification Logic with a natural semantics in which justifications are interpreted as sets of formulas [7]. In addition to its applications in epistemic logic, modal logic and proof theory, LP has been generalized to interactive multi-agent computation [25].

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Studies of justification logics have tended to focus on justified true belief, provability and logical omniscience [19, 3, 2]. Some notions defined for justification logics, such as axiomatically appropriate constant specifications, factivity and logical introspection, reflect this focus [1, 6, 4]. Here we study a logic of justified false belief, Denial Logic DL, defined as JL together with the Denial axiom $t : E \rightarrow \neg E$ and the Evidence Pairing axiom $s : D \wedge t : E \rightarrow [s \& t] : D \wedge E$. DL is the logic of an agent whose justified beliefs are false, who cannot avow his own propositional attitudes or believe tautologies, but who can believe contradictions.

This paper is organized as follows. The first section defines the syntax of Denial Logic and shows that DL cannot justify any of its axioms. We observe that notion of constant specification for JL can lead to inconsistency in DL. Accordingly, we define the notion of negative constant specification.

The second section extends Artemov’s natural semantics for JL to DL. We define the notion of model for DL, prove completeness, and give an inductive construction of models of DL. We apply the inductive construction of models of DL to produce models satisfying negative constant specifications and to prove that the Evidence Pairing axiom is independent of the other axioms of DL.

The third section applies DL to Putnam’s brains in vats where we prove a “Blue Pill” theorem [8]. Here we use justification logic to define a formal notion of coherence suggested by philosophical coherence theories of truth [17].

In the fourth section, we note that a DL agent cannot believe tautologies of propositional logic or avow any of his propositional attitudes. To handle agents that can, we extend DL to $DL \oplus_{JL} LP$, the algebraic fibring of DL with the Logic of Proofs LP constrained by justification logic JL.

In the fifth section we apply $DL \oplus_{JL} LP$ to an agent who denies that climate models indicate anything true, but who allows that CO_2 is a greenhouse gas. The Denial axiom of the DL fragment of $DL \oplus_{JL} LP$ is used to model the agent’s assertion that every indicator produced by a climate model is wrong. This agent cannot provably justify any any scientific conclusion that might follow from the concession that CO_2 is a greenhouse gas.

In the last section we observe that the Blue Pill theorem extends to $DL \oplus_{JL} LP$.

0.1. Related work. Epistemic logics have been used and developed for the analysis of philosophical skepticism. Steiner treats Cartesian skepticism in a normal modal logic and shows that a strong skeptical argument remains if the KK axiom is dropped [41]. Schotch and Jennings propose an alternative to possible-world semantics and define Basic Epistemic Logic, a non-normal modal logic developed to address problems of logical omniscience and skepticism [37]. Schotch defines the proto-epistemic logics, in which the knowledge modality need not distribute over implication [38]. The application here of justification logic to philosophical skepticism appears to be new.

1. DENIAL LOGIC

The language of DL is the same as that of justification logic JL, with the addition of a binary operation $\&$ on justification terms [6]. Symbols of DL include those occurring in justification terms as follows.

justification constants c, c_1, c_2, \dots
justification variables $u, v, w, x, y, z, x_1, x_2, \dots$
binary operations $\cdot, +, \&$
punctuation $[,]$

The set of justification constants, which may be finite or infinite, is specified through what we will call a negative constant specification (see subsection 1.1).

A *justification term* t is an expression of the form

$$t ::= c_i \mid x_j \mid [t + t] \mid [t \cdot t] \mid [t \& t]$$

where c_i is a justification constant and x_j is a justification variable.

Symbols occurring in formulas include justification terms and the following.

propositional variables $A, B, C, \dots, X, Y, Z, A_1, A_2, \dots$
propositional constant \perp
unary connective \neg
binary connectives $\wedge, \vee, \rightarrow$
punctuation $(,), :$

A *formula* A of a DL is an expression of the form

$$A ::= \perp \mid A_i \mid (A \wedge A) \mid (A \vee A) \mid (A \rightarrow A) \mid t : A$$

where t is a justification term and where A_i is a propositional variable. It should be noted that in the formula $t : P$, the justification term t justifies specific propositional syntax: $t : P$ may hold but $t : P \wedge P$ may fail in some model of JL.

The axioms of DL include axioms of classical logic, the Application and Sum axioms of JL, the Denial axiom and the Evidence Pairing axiom:

- (1.1) Application. $s : (P \rightarrow Q) \rightarrow (t : P \rightarrow [s \cdot t] : Q)$
- (1.2) Sum. $s : P \rightarrow [s + t] : P, \quad t : P \rightarrow [s + t] : P$
- (1.3) Denial. $t : P \rightarrow \neg P$
- (1.4) Evidence Pairing. $(s : P \wedge t : Q) \rightarrow [s \& t] : (P \wedge Q)$

in which s and t are justification terms. The Evidence Pairing Axiom is redundant in justification logics with axiomatically complete constant specifications but not in DL (cf. Theorem 18 and the remarks following for definitions).

The rule of inference of DL is *modus ponens* MP. We write $\vdash P$ if P is provable in DL from the axioms and MP.

Denial Logic cannot justify its own Denial Axiom.

Proposition 1. $\text{DL} + \{s : (t : P \rightarrow \neg P)\}$ is inconsistent.

Proof. Suppose that for some justification term s , $\vdash s : (t : P \rightarrow \neg P)$. From the Denial Axiom, $\vdash s : (t : P \rightarrow \neg P) \rightarrow \neg(t : P \rightarrow \neg P)$. By MP, $\vdash \neg(t : P \rightarrow \neg P)$. \square

1.1. Negative constant specifications. A *constant specification* CS for a justification logic L is a set of formulas $e_n : \dots : e_1 : A$ in which e_1, \dots, e_n are justification constants and A is an axiom of L , and which is downward closed: if $e_n : \dots : e_1 : A$ is in CS, then so is $e_{n-1} : \dots : e_1 : A$ [6]. In particular, if $e : A$ is in CS, then so is A . An argument similar to that of Proposition 1 shows that nonempty constant specifications for DL are inconsistent unless they contain no formula of the form $e : P$.

However, DL can have what we call negative constant specifications. A *negative constant specification* NCS for DL is a collection \mathcal{C} of formulas of the form $e_1 : \dots : e_n : P$ or the negation of such a formula, where the e_i are justification constants, such that $\text{DL} + \text{NCS}$ is consistent and where \mathcal{C} is closed under the following rules.

Rule 1. If $e : P \in \mathcal{C}$, then $\neg P \in \mathcal{C}$.

Rule 2. If $\neg e : P \in \mathcal{C}$, then $P \in \mathcal{C}$.

Rule 1 is motivated by the Denial axiom and Rule 2 is consistent with it. If $\text{DL} + \{\neg R\}$ is consistent, then $\{e_1 : R, \neg R\}$ is a negative constant specification.

The choice of a language L for DL together with a negative constant specification (which may be empty) determines we call a *denial logic*.

2. NATURAL SEMANTICS OF DL

We recall Artemov's natural semantics for JL [7]. Let L be a justification logic. Let $\mathbf{2} = \{0, 1\}$ denote the set of truth values, let \mathbf{Var} denote the set of propositional variables of L , let \mathbf{Tm} denote the set of justification terms of L , and let \mathbf{Fm} denote the set of L -formulas. For $X, Y \subseteq \mathbf{Fm}$ we define the set $X \cdot Y$ (of consequents of implications in X and antecedents in Y) by

$$X \cdot Y = \{Q \mid (P \rightarrow Q) \in X \wedge P \in Y\}.$$

Also, we define

$$X \& Y = \{P \wedge Q \mid P \in X \wedge Q \in Y\}.$$

Following Artemov, a *modular model* of JL is a pair of maps, both denoted by $*$, of types $\mathbf{Var} \rightarrow \mathbf{2}$ and $\mathbf{Tm} \rightarrow \mathbf{2}^{\mathbf{Fm}}$ respectively, which satisfy the following relations [7].

$$(1) \quad s^* \cdot t^* \subseteq [s \cdot t]^*$$

$$(2) \quad s^* \cup t^* \subseteq [s + t]^*$$

We define $(t : P)^* = 1$ iff $P \in t^*$. The model $*$ is extended homomorphically to Boolean connectives in the obvious way. We write $\models P$ for $P^* = 1$.

The relation (1) corresponds to the Application axiom (1.1) and the relation (2) corresponds to the Sum Axiom (1.2) of JL [7].

Proposition 2. *Let $*$ be a modular model of JL and let L be a denial logic. The Evidence Pairing axiom of L is satisfied by $*$ if the relation (3) holds.*

$$(3) \quad s^* \& t^* \subseteq [s \& t]^*$$

Proof. Suppose that $*$ is a modular model. If $\models s : P$ and $\models t : Q$ hold, then $P \in s^*$ and $Q \in t^*$, which implies that $P \wedge Q \in s^* \& t^*$. By (3), $P \wedge Q \in [s \& t]^*$, so that $\models [s \& t] : P \wedge Q$. It follows that $\models s : P \wedge t : Q \rightarrow [s \& t] : P \wedge Q$ holds if (3) holds. \square

The image t^* of a justification term is a set of formulas called a *justification set*. Models of DL satisfy an additional property: justification sets of modular models of DL contain only false formulas.

Proposition 3. *Let $*$ be a modular model of DL. For every justification term t ,*

$$(4) \quad t^* \subseteq *^{-1}(0).$$

Proof. $\models t : P \rightarrow \neg P$ iff $t : P^* = 0$ or $P^* = 0$ iff $P \notin t^*$ or $P^* = 0$ iff $t^* \subseteq *^{-1}(0)$. \square

A *modular model* of DL is a modular model $*$ of JL that satisfies (3) and (4). Given a negative constant specification *NCS* DL, a model $*$ *respects NCS* if all formulas of the *NCS* hold in $*$. This is a translation to DL of the analogous notion for constant specifications defined in [6, 7].

Example 4. There are obvious models of DL: any map $* : \mathbf{Var} \rightarrow \mathbf{2}$ can be extended to $\mathbf{Tm} \rightarrow \mathbf{2}^{\mathbf{Fm}}$ by setting $t^* = \emptyset$ for any justification term t ; from there $*$ is extended to \mathbf{Fm} in the obvious way. We call such a modular model a *trivial* model. The Denial axiom is satisfied in all trivial models.

2.1. Soundness and completeness of DL.

Theorem 5. *DL + NCS $\vdash \varphi$ if and only if $\models \varphi$ holds in every model of DL respecting NCS.*

Proof. The soundness DL follows from Proposition 3 and Theorem 1 of [7]. The proof of completeness is the same as in [7]. \square

2.2. Inductive construction of models. We give an inductive construction of all modular models of DL, depending on a valuation of propositional variables and a Boolean valued-functional.

This construction of a nontrivial modular model for DL will extend a map $* : \mathbf{Var} \rightarrow \mathbf{2}$ simultaneously to $\mathbf{Tm} \rightarrow \mathbf{2}^{\mathbf{Fm}}$ and to $* : \mathbf{Fm} \rightarrow \mathbf{2}$. These models will be parametrized by a Boolean-valued functional of type $F : \mathbf{2}^{\mathbf{Var}} \times \tau^3 \rightarrow \mathbf{2}$, where the ordinal τ will be defined.

Since \mathbf{Tm} is inductively defined, there exist well orderings on \mathbf{Tm} so that if $\gamma_0, \dots, \gamma_n$ are ordinals, $t_{\gamma_i} \in \mathbf{Tm}$ and $t_{\gamma_n} = F(t_{\gamma_0}, \dots, t_{\gamma_{n-1}})$ then $\gamma_0, \dots, \gamma_{n-1} < \gamma_n$. Fix such a well ordering on \mathbf{Tm} . Likewise, \mathbf{Fm} is constructed inductively from \mathbf{Var} and \mathbf{Tm} , so a well ordering can be defined on \mathbf{Fm} so that for ordinals $\gamma_0, \dots, \gamma_{n-1}, \beta_0, \dots, \beta_m$, if $P_{\beta_j} \in \mathbf{Fm}$ and

$P_{\beta_m} = G(t_{\gamma_0}, \dots, t_{\gamma_{n-1}}; P_{\beta_0}, \dots, P_{\beta_{m-1}})$, then for $0 \leq i < n$ and $0 \leq j < m$, $\max(\gamma_i, \beta_j) < \beta_m$. There may exist ordinals β for which P_β is undefined; we assume that a given case below holds only when relations occurring in them are defined and hold—this applies to the $t_\gamma \in \mathbf{Tm}$ as well.

The well ordering of \mathbf{Fm} implies that $\mathbf{Fm} = \bigcup_{\alpha < \tau} \mathbf{Fm}_\alpha$, where $\mathbf{Fm}_\alpha = \{P_\beta : \beta < \alpha\}$, and where τ is the order type of the chosen well ordering of \mathbf{Fm} . For $\gamma < \alpha < \tau$ and $t_\gamma \in \mathbf{Tm}$, we will inductively define the set $t_{\gamma, \alpha}^* \subseteq \mathbf{Fm}_\alpha$ and the satisfaction relation $*_\alpha : \mathbf{Fm}_\alpha \rightarrow \mathbf{2}$.

At the end of the construction we set $t_\gamma^* = \bigcup_{\alpha < \tau} t_{\gamma, \alpha}^*$ and likewise extend the $*_\alpha$ to \mathbf{Fm} . We proceed by induction on $\alpha < \tau$.

Suppose $\alpha = 0$. Set all $t_{\gamma, 0}^* = \emptyset$ and let $*_0$ be the empty function $\mathbf{Fm}_0 \rightarrow \mathbf{2}$.

Suppose $\alpha > 0$. The inductive hypothesis is that $*_\sigma$ is defined on \mathbf{Fm}_σ for $\sigma < \alpha$ and that $t_{\gamma, \sigma}^*$ is defined for $\gamma, \sigma < \alpha$.

If α is a limit ordinal, then set $t_\gamma^* = \bigcup_{\beta < \alpha} t_{\gamma, \beta}^*$ and let $*_\alpha$ be the union of the $*_\beta$ for $\beta < \alpha$. Otherwise, α is a successor ordinal and the remaining cases apply. For Boolean connectives we show \rightarrow only as this is needed to verify the Application axiom; the cases of conjunction \wedge , disjunction \vee and negation \neg follow the same pattern and are not shown.

Let $\beta, \gamma < \alpha$.

Case 1. P_β is in \mathbf{Var} .

1a) Define

$$t_{\gamma, \alpha}^{*, \beta} = \begin{cases} \{P_\beta\}, & \text{if } P_\beta^* = 0 \wedge F(*, \alpha, \beta, \gamma) = 1; \\ \emptyset, & \text{otherwise.} \end{cases}$$

1b) Set $P_\beta^{*\alpha} = P_\beta^*$.

Case 2. $P_\beta = P_\mu \rightarrow P_\nu$ for $\mu, \nu < \alpha$. The well ordering on \mathbf{Fm} ensures that $\mu, \nu < \beta < \alpha$. By the inductive hypothesis, the partial satisfaction relation $*_\beta$ is defined on $\mathbf{Fm}_{\max(\mu, \nu)}$ and hence on P_μ and P_ν .

2a) Define

$$t_{\gamma, \alpha}^{*, \beta} = \begin{cases} \{P_\beta\}, & \text{if } P_\mu^{*\beta} = 1 \wedge P_\nu^{*\beta} = 0 \wedge F(*, \alpha, \beta, \gamma) = 1; \\ \emptyset, & \text{otherwise.} \end{cases}$$

2b) Define

$$P_\beta^{*\alpha} = \begin{cases} 0, & \text{if } P_\mu^{*\beta} = 1 \wedge P_\nu^{*\beta} = 0; \\ 1, & \text{otherwise.} \end{cases}$$

Case 3. $P_\beta = t_\gamma : P_\delta$ for $\gamma, \delta < \alpha$. Again, $\gamma, \delta < \beta < \alpha$ and by the inductive hypothesis, $P_\delta^{*\beta}$ is defined.

3a) Define

$$t_{\gamma, \alpha}^{*, \beta} = \begin{cases} \{P_\delta\}, & \text{if } P_\delta^{*\beta} = 0 \wedge F(*, \alpha, \beta, \gamma) = 1; \\ \emptyset, & \text{otherwise.} \end{cases}$$

3b) Define

$$P_{\beta}^{*\alpha} = \begin{cases} 1, & \text{if } P_{\delta}^{*\beta} = 0 \wedge F(*, \alpha, \beta, \gamma) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Case 4. None of the above (P_{β} is undefined). Do nothing.

Prologue. At the end of stage α , we define for $\gamma < \alpha$,

$$t_{\gamma, \alpha}^* = \bigcup_{\beta < \alpha} t_{\gamma, \alpha}^{*, \beta} \cup \bigcup_{\sigma < \alpha} t_{\gamma, \sigma}^*$$

Before proceeding to stage $\alpha + 1$, we may need to update the justification set $t_{\gamma, \alpha}^*$ to ensure that the Sum axiom (1.2) and the Evidence Pairing axiom (1.4) continue to hold. For each $\gamma < \alpha$ there are three cases.

Case a) There exist $\mu, \nu < \gamma$ such that $t_{\gamma} = t_{\mu} + t_{\nu}$. If $t_{\mu, \alpha}^* \cup t_{\nu, \alpha}^* \subseteq t_{\gamma, \alpha}^*$ do nothing; otherwise redefine

$$t_{\gamma, \alpha}^* := t_{\mu, \alpha}^* \cup t_{\nu, \alpha}^* \cup \bigcup_{\beta < \alpha} t_{\gamma, \alpha}^{*, \beta} \cup \bigcup_{\sigma < \alpha} t_{\gamma, \sigma}^*.$$

Case b) There exist $\mu, \nu < \gamma$ such that $t_{\gamma} = t_{\mu} \& t_{\nu}$. If $t_{\mu, \alpha}^* \& t_{\nu, \alpha}^* \subseteq t_{\gamma, \alpha}^*$ do nothing; otherwise redefine

$$t_{\gamma, \alpha}^* := (t_{\mu, \alpha}^* \& t_{\nu, \alpha}^*) \cup \bigcup_{\beta < \alpha} t_{\gamma, \alpha}^{*, \beta} \cup \bigcup_{\sigma < \alpha} t_{\gamma, \sigma}^*.$$

Case c) None of the above. This includes the possibility that t_{γ} is undefined. Do nothing.

(End of construction 2.2).

Proposition 6. *For each Boolean-valued functional $F : \mathbf{2}^{\mathbf{Var}} \times \tau^3 \rightarrow \mathbf{2}$ and for each valuation $* : \mathbf{Var} \rightarrow \mathbf{2}$, the construction 2.2 yields a modular model of DL.*

Proof. We show that conditions (1), (2) and (3) are satisfied. Note that at each stage α only formulas that evaluate to 0 (false) are enumerated into the image of any justification term. Hence (3) is satisfied, and by Proposition (3), the Denial axiom (1.3) holds.

For (1), note that for any $s, t \in \mathbf{Tm}$, $s^* \cdot t^* = \emptyset$. Otherwise there exist $P, Q \in \mathbf{Fm}$ such that $Q \in s^* \cdot t^*$, $P \rightarrow Q \in s^*$ and $P \in t^*$. But then $(P \rightarrow Q)^* = 0$, which forces $P^* = 1$ and $Q^* = 0$. But since $P \in t^*$, by construction, $P^* = 0$. This is a contradiction. Therefore (1) holds vacuously, and the Application axiom (1.1) holds.

The Prologue at stage α of Construction (2.2) ensures that the Sum Axiom (1.2) and the Evidence Pairing Axiom (1.4) hold. \square

2.3. Examples of modular models.

Example 7. Trivial modular models are obtained by setting $F(*, \alpha, \beta, \gamma) = 0$ for all valuations $* : \mathbf{Var} \rightarrow \mathbf{2}$ and for all ordinals $\alpha, \beta, \gamma < \tau$.

Example 8. Setting $F = 1$ yields modular models such that $t^* = *^{-1}(0)$ for every justification term t . These models are maximal.

Example 9. This example will illustrate that the inclusion in (2) can be strict [7]. Define the functional $F : \mathbf{2}^{\mathbf{Var}} \times \tau^3 \rightarrow \mathbf{2}$ by

$$F(*, \alpha, \beta, \gamma) = \begin{cases} 1, & \text{if } t_\gamma \text{ contains a } +; \\ 0, & \text{otherwise.} \end{cases}$$

Let $*$ be a valuation and suppose that the formula P is false in the model determined by F and $*$. Then $\not\models [x + y] : P \rightarrow x : P \vee y : P$.

This syntactical criterion ensures that for any false formula P , if the justification t has sufficient complexity, then $t : P$ is true.

Example 10. Every modular model of **DL** arises through the choice of a pair (F, v) consisting of a functional F and a valuation $v : \mathbf{Var} \rightarrow \mathbf{2}$. Given a negative constant specification there is a choice of a functional F that realizes it. For α large enough and for $P_\beta = e_1 : \dots : e_n : P$ in the specification, where $e_1 = t_\gamma$, set $F(*, \alpha, \beta, \gamma) = 1$. If $P_\beta = \neg e_1 : \dots : e_n : P$, then set $F(*, \alpha, \beta, \gamma) = 0$.

Example 11. Let DL° be denial logic without the Pairing axiom 1.3 and the binary operation $\&$ on justification terms. The completeness theorem holds for this logic. Also, the construction above goes through without mention of the operation $\&$ or the Pairing axiom to yield an inductive construction of models of DL° . Let $NCS = \{a : A, \neg A, b : B, \neg B\}$ and assume that $\text{DL}^\circ + NCS$ is consistent. We may use the construction and the procedure of the previous example to build a model of $\text{DL}^\circ + NCS$ such that for each justification term t , $t^* \subseteq \{A, B\}$. It follows that there is no justification term s such that $\models s : A \wedge B$ (otherwise $A \wedge B \in s^*$, contrary to the construction). Hence there is no justification term s such that $\vdash_{\text{DL}^\circ + NCS} s : A \wedge B$. This shows that the Evidence Pairing axiom is independent of the other axioms of **DL**.

3. PHILOSOPHICAL INTERPRETATION OF DL

Artemov has characterized Justification Logic **JL** with an empty constant specification as the “logic of general (not necessarily factive) justifications for an absolutely skeptical agent for whom no formula is provably justified” [6]. Denial Logic **DL** models an agent whose justified beliefs are false, who cannot avow his own propositional attitudes, who is capable of believing logical contradictions and for whom even tautologies of classical logic cannot be justified. The soundness and completeness of **DL** and the number of its

nontrivial modular models suggest that DL is suitable as a logic of philosophical skepticism that goes beyond the skeptical challenges of Descartes' *Meditations* and Putnam's brains in vats [18, 33].

In his *First Meditation*, Descartes writes that he can still reason about arithmetic and geometry even if he is dreaming: "For whether I am awake or asleep, two and three added together are five, and a square has no more than four sides. It seems impossible that such transparent truths should incur any suspicion of being false" [18]. In our applications of DL below, our agents will stop short of believing contradictions and disavowing their own beliefs.

In *Reason, Truth and History*, Hilary Putnam presented a modern formulation of the skeptical challenge posed by the evil genius of Descartes' *Meditations* [33]. Putnam's argument that we could not be brains in a vat provoked a vigorous philosophical response, e.g. [33, 29, 40, 22, 13, 16, 10, 14, 43, 42, 12, 21, 31, 36, 47, 32, 24, 11]. For Putnam, the thesis that we are, have always been, and will always be brains in a vat is self-refuting on semantic grounds. The argument remains controversial.

3.1. Brains in vats under DL. We interpret Putnam's thought experiment so that the logic of belief of our brain in a vat concerning its sensory experience is denial logic DL. In this interpretation, the world consists of a bio-computing facility supplying electrical impulses to a community of brains in vats with nutrients and appropriate cabling to the computing facility. The programming of the facility determines the sense experiences of the envatted brains as its simulation runs [33]. We assume that logic of belief of an envatted brain of its sense experience is given by a choice of language for DL together with the choice of a negative constant specification CS of the justified false beliefs the brain holds about its experience.

More explicitly, we assume that for any sense experience the computing facility induces within an envatted brain, there are one or more formulas E of DL asserting something that the brain believes about that experience (e.g., "I am not a brain in a vat"). Also, we assume that for every such formula E that the envatted brain believes for some reason indicated by its experience, there is a justified formula $s : E$ of CS with justification term s , such that in every model $*$ of $DL + CS$, the interpretation s^* of s in the model $*$ contains the formulas representing the account the brain would provide of its experience, were the brain to attempt to justify its belief (e.g., "I am walking outside", "the air is cool", "I hear the cellphone ringing" and so on).

Following Putnam's thought experiment, we will assume that the sentence "I am not a brain in a vat" is represented in $DL + CS$ as $s : E$ for some justification term s . In $DL + CS$, $\neg E$ holds; that is, the brain is indeed a brain in a vat. In general we assume that $DL + CS$ can formally represent the epistemic state of an envatted brain as the computing facility determines the brain's sense experience, at least to one order of belief. Beliefs about beliefs will be addressed in the sequel.

Our intention is that $DL + CS$ formalizes the epistemic situation of the envatted brain, as expressed by Ludwig in this passage [27, p.35–36].

[...] the brain in the vat is thinking ‘There’s a tree’ and means what we do by that, and fails to be, in one sense, thinking about a tree simply because his assumption that he is in perceptual contact with a tree is false. The brain in the vat, then, far from having mostly true beliefs in virtue of its not being in causal contact with trees and tables and chairs and the like, and so not thinking about such things, has mostly false beliefs precisely because he fails, when talking about things about him, to be thinking about trees and tables and chairs and so on.

Given $DL + CS$, we use the knowledge extraction operator OK of Artemov and Kuznets to derive a new logical system with an interpretation $*$ such that if $\{s : E, \neg E\} \subseteq CS$ (e.g., where $s : E$ is “I am not a brain in a vat for reason s ”), then E will be true in $*$. If L is a justification logic and if CS is a (negative) constant specification, we define the knowledge extraction operator OK as follows [2].

$$OK(CS) := \{E : \exists t \in \mathbf{Tm}, \vdash_{L+CS} t : E\}.$$

A negative constant specification CS for DL is *coherent* if for every justification term t such that $\vdash_{DL+CS} t : E$, there exists a model $*$ of JL such that $\models E$. By JL we mean justification logic with the same language as $DL + CS$, but without the specification.

Proposition 12. *Suppose that CS is a coherent constant specification for DL . Then every finite subcollection of $OK(CS)$ is satisfiable in some modular model of JL .*

Proof. Suppose otherwise. Then there is a finite sequence $t_1 : E_1, \dots, t_n : E_n$ of justified formulas with $E_i \in OK(CS)$ such that $\vdash_{DL+CS} t_i : E_i$ for $1 \leq i \leq n$, and such that $((\dots (E_1 \wedge E_2) \wedge \dots) \wedge E_n)$ is unsatisfiable in JL . By logic,

$$\vdash_{DL+CS} ((\dots (t_1 : E_1 \wedge t_2 : E_2) \wedge \dots) \wedge t_n : E_n)$$

It follows from the Evidence Pairing axiom1.4 that there exists a justification term $t = [[\dots [t_1 \& t_2] \& \dots] \& t_n]$ such that

$$\vdash_{DL+CS} t : ((\dots (E_1 \wedge E_2) \wedge \dots) \wedge E_n).$$

Since CS is coherent, there is a model $*$ of JL in which $\models ((\dots (E_1 \wedge E_2) \wedge \dots) \wedge E_n)$. This is a contradiction. \square

Theorem 13 (Blue Pill Theorem). *$JL + OK(CS)$ has a model $*$.*

Proof. By Proposition 12 and the compactness of JL . Compactness follows from the completeness theorem for JL [7]. \square

Under the assumption that the collection of justified beliefs of an envatted brain is coherent, there exists a model $*$ in which “I am not a brain in a vat” comes out true, as well as all of the other justified false assertions of the negative constant specification CS .

The JL model $*$ was derived from Proposition 12 under the assumption that $DL + CS$ is consistent (which implies that it has a model) and that CS is coherent. The mathematical argument makes no reference to Putnam’s causal theory of reference [33]. It does not offer much assurance that we are not brains in a vat, however.

Proposition 12 and Corollary 13 hold in more general justification logics that satisfy an evidence pairing property. In general we have the following.

Corollary 14. *Suppose that CS is a coherent constant specification for a justification logic L . Suppose further that whenever $\vdash_{L+CS} r : A \wedge s : B$ there exists a justification term t such that $\vdash_{L+CS} t : A \wedge B$. Then every finite subcollection of $OK(CS)$ is satisfiable in some modular model of JL with the same language as L .*

Corollary 15. *$JL + OK(CS)$ has a model $*$.*

4. MULTI-MODAL DENIAL LOGIC

Crispin Wright’s discussion of propositional attitudes in *On Putnam’s Proof that we are not Brains-in-a-vat* suggests why DL is not suitable for modeling beliefs about belief [48, p. 79].

It is part of the way we ordinarily think about self-consciousness that we regard the contents of a subject’s contemporary propositional attitudes as something which, to use the standard term of art, they can *avow*—something about which their judgments are credited with a strong, though defeasible authority which does not rest on reasons or evidence.

If knowledge is true justified belief, DL cannot model agents that know anything. Nevertheless, DL is one starting point for logical models of skeptical thought experiments. To handle agents who can believe theorems of classical logic and verify them, and who can avow their own propositional attitudes, we can extend DL to $DL \oplus_{JL} LP$, the algebraic fibring of Denial Logic DL with the Logic of Proofs LP constrained by justification logic JL. This is a pushout construction, first described in [39].¹

$$\begin{array}{ccc} JL & \rightarrow & LP \\ \downarrow & & \downarrow \\ DL & \rightarrow & DL \oplus_{JL} LP \end{array}$$

¹More generally, we suggest that the Humean distinction between “matters of fact” and “relations of ideas” may be modeled by the algebraic fibring of two justification logics: one appropriate for matters of fact and one appropriate to relations of ideas (e.g., analytic propositions) [23].

We indicate the construction in our case, which amounts to the fusion of modal propositional Hilbert calculi.

Syntax and semantics of $\text{DL} \oplus_{\text{JL}} \text{LP}$. The syntax of $\text{DL} \oplus_{\text{JL}} \text{LP}$ amends that of DL as follows. Justification constants and variables are signed and have the form c_i^σ, x_i^σ where $\sigma \in \{+, -\}$. The signed justification constant c_i^+ (variable x_i^+) is *positive*, and the signed justification constant c_i^- (variable x_i^-) is *negative*. A *positive (negative) justification term* of $\text{DL} \oplus_{\text{JL}} \text{LP}$ is an expression of the form

$$t^\sigma ::= c_i^\sigma \mid x_j^\sigma \mid [t^\sigma + t^\sigma] \mid [t^\sigma \cdot t^\sigma] \mid [t^- \& t^-] \mid !t^+$$

where $\sigma \in \{+, -\}$, t^+ is positive, and t^- is negative. The set \mathbf{Tm} of justification terms is the disjoint union of the set \mathbf{Tm}^+ positive justification terms and the set \mathbf{Tm}^- of negative justification terms.

The definition of a formula of $\text{DL} \oplus_{\text{JL}} \text{LP}$ amends the definition of a formula of denial logic DL as follows. A *formula* A of DL^\pm is an expression of the form

$$A ::= \perp \mid A_i \mid (A \wedge A) \mid (A \vee A) \mid (A \rightarrow A) \mid t^+ : A \mid t^- : A$$

where t^+ is a positive justification term, t^- is a negative justification term, and where A_i is a propositional variable. The axioms of $\text{DL} \oplus_{\text{JL}} \text{LP}$ are the same as DL and LP with the proviso that the Denial and Evidence Pairing axioms are restricted to negatively justified formulas, factivity and introspection are restricted to positively justified formulas, and the signs of the terms appearing in the Sum and Application axioms must be the same. Equivalently, the following axioms are those of $\text{DL} \oplus_{\text{JL}} \text{LP}$.

(4.1) Application. $s^\sigma : (P \rightarrow Q) \rightarrow (t^\sigma : P \rightarrow [s^\sigma \cdot t^\sigma] : Q)$

(4.2) Sum. $s^\sigma : P \rightarrow [s^\sigma + t^\sigma] : P, \quad t^\sigma : P \rightarrow [s^\sigma + t^\sigma] : P \vdash$

(4.3) Denial. $s^- : P \rightarrow \neg P$

(4.4) Pairing. $(s^- : P \wedge t^- : Q) \rightarrow [s^- \& t^-] : (P \wedge Q)$

(4.5) Positive Factivity. $t^+ : P \rightarrow P$

(4.6) Positive Introspection. $t^+ : P \rightarrow !t^+ : t^+ : P$

where $\sigma \in \{+, -\}$, s^- is negative, and t^+ is positive. The rule of inference in $\text{DL} \oplus_{\text{JL}} \text{LP}$ is modus ponens.

Proposition 16. $\text{DL} \oplus_{\text{JL}} \text{LP} \vdash \neg(s^- : P \wedge t^+ : P)$, where s^- is negative, t^+ is positive and where P is a formula.

Proof. Immediate from Axioms 4.3 and 4.5. \square

Constant specifications for $\text{DL} \oplus_{\text{JL}} \text{LP}$ generalize those of JL and DL. A *constant specification* for $\text{DL} \oplus_{\text{JL}} \text{LP}$ is a collection \mathcal{C} of formulas of the form $e_1 : \dots : e_n : P$ and negations of such formulas, where the e_i are justification constants, such that $(\text{DL} \oplus_{\text{JL}} \text{LP}) + \mathcal{C}$ is consistent and where \mathcal{C} is closed under the following rules.

Rule 1. If $t : P \in \mathcal{C}$ and t is positive, then $P \in \mathcal{C}$.

Rule 2. If $s : P \in \mathcal{C}$ and s is negative, then $\neg P \in \mathcal{C}$.

Rule 3. If $\neg t : P \in \mathcal{C}$ and t is positive, then $\neg P \in \mathcal{C}$.

Rule 4. If $\neg s : P \in \mathcal{C}$ and s is negative, then $P \in \mathcal{C}$.

This definition is motivated as follows. A set \mathcal{C} consisting of formulas only of the form $e_1 : \dots : e_n : P$ with all e_i positive is just a constant specification in the usual sense. Rule 1 is forced by the Positive Factivity axiom (4.5), and Rule 3 is consistent with it. Dually, Rule 2 is forced by signed denial, and Rule 4 is consistent with it.

A *modular model* of $\text{DL} \oplus_{\text{JL}} \text{LP}$ is a pair of maps, both denoted by $*$, of types $\mathbf{Var} \rightarrow \mathbf{2}$ and $\mathbf{Tm}^- \cup \mathbf{Tm}^+ = \mathbf{Tm} \rightarrow \mathbf{2}^{\mathbf{Fm}}$ that satisfy the following conditions.

- (5) $(s^\sigma)^* \cdot (t^\sigma)^* \subseteq [s^\sigma \cdot t^\sigma]^*$
- (6) $(s^\sigma)^* \cup (t^\sigma)^* \subseteq [s^\sigma + t^\sigma]^*$
- (7) $(s^-)^* \& (t^-)^* \subseteq [s^- \& t^-]^*$
- (8) $(t^-)^* \subseteq (*|\mathbf{Tm}^-)^{-1}(0)$
- (9) $(t^+)^* \subseteq (*|\mathbf{Tm}^+)^{-1}(1)$
- (10) $P \in (t^+)^* \rightarrow t^+ : P \in (!t^+)^*$

where $s^\sigma, t^\sigma \in \mathbf{Tm}^\sigma$, $\sigma \in \{+, -\}$.

If CS is a constant specification for $\text{DL} \oplus_{\text{JL}} \text{LP}$, then a model $*$ *respects* CS if all formulas of the CS hold in $*$ [6, 7]. The analog of Theorem 5, the soundness and completeness theorem, holds for $\text{DL} \oplus_{\text{JL}} \text{LP}$.

Theorem 17. $\text{DL} \oplus_{\text{JL}} \text{LP}$ is complete.

Proof. Suppose that φ is a formula of $\text{DL} \oplus_{\text{JL}} \text{LP}$ such that

$$(11) \quad \not\vdash_{\text{DL} \oplus_{\text{JL}} \text{LP}} \varphi.$$

Let L be a justification logic, and let $\mathbf{Fm}(L)$ denote the set of formulas of L . Let V be a set of new propositional variables $X_{s^-:\psi}$ for each negatively justified formula $s^- : \psi$ of $\text{DL} \oplus_{\text{JL}} \text{LP}$. Let L' denote the justification logic obtained from the LP fragment of $\text{DL} \oplus_{\text{JL}} \text{LP}$ by adjoining the set V of new variables and closing under Boolean connectives and positive justification terms. If $s : \psi \in \mathbf{Fm}(L')$, then s is a positive justification term.

Define by induction a transformation $T : \mathbf{Fm}(\text{DL} \oplus_{\text{JL}} \text{LP}) \rightarrow \mathbf{Fm}(L')$ as follows.

$$\begin{aligned} T(E) &= E \text{ if } E \text{ is a propositional variable;} \\ T(\neg E) &= \neg T(E); \\ T(D \circ E) &= T(D) \circ T(E) \text{ if } \circ \in \{\wedge, \vee, \rightarrow\}; \\ T(s^+ : E) &= s^+ : T(E) \text{ for } s^+ \text{ positive;} \\ T(s^- : E) &= X_{s^-:T(E)} \text{ for } s^- \text{ negative.} \end{aligned}$$

The logic L' together with the T -images of the axioms of $\text{DL} \oplus_{\text{JL}} \text{LP}$ and with Modus Ponens as the rule of inference is a justification logic satisfying the axioms of the Logic of Proofs. This is because the T -images of classical logic axioms in $\text{DL} \oplus_{\text{JL}} \text{LP}$ are precisely the substitution instances of those axioms

in $\mathbf{Fm}(L')$; and similarly the T -images of the application, factivity, sum and proof checker axioms of the LP fragment of $\text{DL} \oplus_{\text{JL}} \text{LP}$ are precisely the substitution instances of those axioms in $\mathbf{Fm}(L')$. Hence for $\psi \in \mathbf{Fm}(\text{DL} \oplus_{\text{JL}} \text{LP})$, $\vdash_{\text{DL} \oplus_{\text{JL}} \text{LP}} \psi \Leftrightarrow \vdash_{L'} T(\psi)$. It follows from (11) that $\not\models T(\varphi)$. By the completeness theorem for LoP , there is a modular model $*'$ of L' such that $*' \not\models T(\varphi)$ [26].

Define a model $*$ of $\text{DL} \oplus_{\text{JL}} \text{LP}$ from $*'$ as follows. If ψ is a propositional variable of $\text{DL} \oplus_{\text{JL}} \text{LP}$, take $*\psi = *'\psi$. If t^+ is a positive justification term, take $(t^+)^* = (t^+)^{*'}$. If s^- is a negative justification term, define $(s^-)^*$ to be the set such that for each $\psi \in \mathbf{Fm}(\text{DL} \oplus_{\text{JL}} \text{LP})$,

$$\psi \in (s^-)^* \Leftrightarrow *' \models E_{s^-:T(\psi)}.$$

By induction, if ψ is a formula of $\text{DL} \oplus_{\text{JL}} \text{LP}$, then

$$(12) \quad * \models \psi \Leftrightarrow *' \models T(\psi).$$

For example, $* \models t^- : \psi \Leftrightarrow \psi \in (s^-)^* \Leftrightarrow *' \models T(t^- : \psi)$.

Moreover, $*$ satisfies conditions (5) through (9) of subsection 5. For example, the T -image of the Denial axiom ensures that $*$ satisfies condition (4) above. If $* \models s^- : \psi$ holds, then so does $*' \models E_{s^-:T(\psi)}$. Since the T -image of the Denial axiom holds, $*' \models E_{s^-:T(\psi)} \rightarrow \neg T(\psi)$, and hence $*' \models \neg T(\psi)$ holds. But by (12), $* \models \neg\psi$, which yields (4). The other cases are similar. Hence $*$ is a model of $\text{DL} \oplus_{\text{JL}} \text{LP}$ in which $* \not\models \varphi$. \square

4.1. Internalization in $\text{DL} \oplus_{\text{JL}} \text{LP}$. A constant specification CS for a justification logic JL is *axiomatically appropriate* if for each axiom A of JL, there is a justification constant e_1 such that $e_1 : A$ is in CS , and if CS is downward closed (cf. subsection 1.1) [6]. The Logic of Proofs LP can internalize its deductions. More generally, we have the following.

Theorem 18 (Theorem 1, [6]). *For each axiomatically appropriate constant specification CS , $\text{JL} + CS$ enjoys the Internalization Property:*

If $\vdash F$, then $\vdash p : F$ for some justification term p .

Recall that DL with a nontrivial constant specification is inconsistent. *A fortiori*, DL with an axiomatically appropriate constant specification is inconsistent. This fact motivated our definition of negative constant specifications for DL and our generalization of negative constant specifications to $\text{DL} \oplus_{\text{JL}} \text{LP}$. Accordingly, the definition of axiomatically appropriate constant specifications in $\text{DL} \oplus_{\text{JL}} \text{LP}$ must be restricted to constant specifications CS with positive justification constants applied to the axioms of $\text{DL} \oplus_{\text{JL}} \text{LP}$. With this restriction, $\text{DL} \oplus_{\text{JL}} \text{LP} + CS$ satisfies the internalization property.

We also note that in justification logics with axiomatically appropriate constant specifications, the Evidence Pairing axiom is unnecessary. In such a system, there is a justification constant a such that

$$(13) \quad a : (P \rightarrow (Q \rightarrow (P \wedge Q)))$$

holds. If $s : P$ holds, then by the Application axiom and modus ponens we have $[a \cdot s] : (B \rightarrow (P \wedge Q))$. And if $t : Q$ holds, application and modus ponens gives us $[[a \cdot s] \cdot t] : (P \wedge Q)$. However, (13) is inconsistent with DL.

5. ANTHROPOGENIC GLOBAL WARMING DENIAL IN $DL \oplus_{JL} LP$

Earth scientists often speak of the indicators produced by their biogeophysical models [45, 46, 35]. The scientific modeling, remote sensing and economic literature is replete with this usage: ecologists speak of indicators of ecosystem stress; meteorologists speak of atmospheric indicators of climate change, economists speak of economic indicators and so on [28, 34, 44]. The scientific vocabulary of indicators of biogeophysical models suggests reading $t : E$ in JL as “ t indicates E .” This reading is more natural for our applications than “term t is justification for E .”

The logic $DL \oplus_{JL} LP$ can model an agent who believes that climate models are as “close to falsification [...] as mathematically possible,” but who allows that CO_2 is a greenhouse gas [15]. For this agent, this the only scientific statement pertaining to global warming that can be justified. We think of the DL fragment of $DL \oplus_{JL} LP$ as the logic of indicators of some climate model. The first assertion is expressed by the Denial axiom, which holds in the DL fragment of $DL \oplus_{JL} LP$. For example, if E is some statement about the environment, such as “global warming is accelerating,” then $t : E$ is the statement that t is indication of some climate model that E . The second statement is positively justified in the LP fragment of $DL \oplus_{JL} LP$ as $s : C$, where s is a positive justification term.

Proposition 19. *In $DL \oplus_{JL} LP$, if $s : C$ with s positive, and if $t : E$ with t negative, then there is no positive justification term j such that $j : (C \rightarrow E)$.*

Proof. Suppose there exists such a j . The Positive Application axiom asserts that for j, s positive justification terms, $[j \cdot s]$ is also positive, and

$$(14) \quad j : (C \rightarrow E) \rightarrow (s : C \rightarrow [j \cdot s] : E).$$

By MP twice, $[j \cdot s] : E$. This together with the assumption $t : E$ contradicts Proposition 16. \square

The agent asserts that the only thing scientists can justify is that CO_2 is a greenhouse gas. This is modeled by adjoining to $DL \oplus_{JL} LP$ the formulas $c : C$ and C , where c is a positive justification constant. In effect, the agent accepts $c : C$ and C as axioms. However, “climate models are as close to falsification [...] as is mathematically possible,” which in $DL \oplus_{JL} LP$ is represented as the Denial axiom in the DL fragment. In this context, in which the DL fragment of $DL \oplus_{JL} LP$ models the logic of indicators of some climate model, the Denial axiom asserts that every indicator produced by that climate model is wrong. This is enough to rule out any scientific conclusion that might follow from the concession that CO_2 is a greenhouse gas.

One can say more about the epistemic state of the agent in a larger system, $DL \oplus_{JL} JL \oplus_{JL} LP$, in which the middle summand can represent justified beliefs that may be true or false. In this case the agent believes that the denial axiom holds for climate models. This is modeled in $DL \oplus_{JL} \oplus_{JL} LP$ by adjoining $e : (s^- : E \rightarrow \neg E)$ where e is a “neutral” justification constant from the middle JL summand.

Also, the agent believes that he knows that the Denial axiom holds for climate models. This is expressed by the formula $e : t^+ : (s^- : E \rightarrow \neg E)$, where t^+ is a positive justification constant.

6. ENVATTED BRAINS UNDER $DL \oplus_{JL} LP$

We augment the situation and logical apparatus of subsection 3.1 by stipulating that the logic of justified belief of the computer generated sensory experiences of the envatted brains is definable within the DL fragment of the logic $DL \oplus_{JL} LP$. Further, this logic can model brains that have positively justified beliefs about their negatively justified beliefs about their sense experience. An envatted brain may correctly believe that it believes that its sensory experience indicates that it is an embodied human (and not a brain confined to a vat). In $DL \oplus_{JL} LP$ this epistemic state is expressed as a formula of the form $t^+ : s^- : E$, in which t^+ represents the positively justified belief that the negatively justified belief represented by s^- indicates E . From $t^+ : s^- : E$ Positive Factivity yields $s^- : E$, and from this Denial yields $\neg E$. We do not stipulate that the Denial axiom is positively justified. The analogs of Proposition 12 and the Blue Pill Theorem 13 hold in $DL \oplus_{JL} LP$.

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